Supplement to the paper "Singular vector distribution of sample covariance matrices "

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Abstract

This supplementary material contains the proofs of Lemma 3.4, 3.8 and 4.4 of the paper.

First of all, we will follow the basic approach of [2, Lemma 6.1] to prove Lemma 3.4, which compares the sharp counting function with its delta approximation smoothed on the scale $\tilde{\eta}$.

Proof of Lemma 3.4. Recall (3.40) of the paper, we have $\tilde{\eta} \ll t \ll E_U - E^- \leq \frac{7}{2} N^{-2/3+\epsilon}$. Furthermore, for $x \in \mathbb{R}$, we have

$$|\mathcal{X}_E(x) - \mathcal{X}_E * \vartheta_{\tilde{\eta}}(x)| = \left| (\int_{\mathbb{R}} \mathcal{X}_E(x) - \int_{E^- - x}^{E_U - x}) \vartheta_{\tilde{\eta}}(y) dy \right|. \tag{S1}$$

Denote $d(x) := |x - E^-| + \tilde{\eta}$ and $d_U(x) := |x - E_U| + \tilde{\eta}$, we need the following bound to estimate (S1).

Lemma 0.1. *There exists some constant $C > 0$, such that*

$$|\mathcal{X}_E(x) - \mathcal{X}_E * \vartheta_{\tilde{\eta}}(x)| \leq C\tilde{\eta} \left[\frac{E_U - E^-}{d_U(x)d(x)} + \frac{\mathcal{X}_E(x)}{(d_U(x) + d(x))} \right].$$

Proof. When $x > E_U$, we have

$$|\mathcal{X}_E(x) - \mathcal{X}_E * \vartheta_{\tilde{\eta}}(x)| = \tilde{\eta} \left| \int_{x-E_U}^{x-E^-} \frac{1}{\pi(y^2 + \tilde{\eta}^2)} dy \right| = \frac{\tilde{\eta}}{\pi} \left[\int_{x-E_U}^{x-E^-} \frac{1}{(y+\tilde{\eta})^2} + \frac{2\tilde{\eta} y}{(y^2+\tilde{\eta}^2)(y+\tilde{\eta})^2} dy \right]$$
$$\leq C\tilde{\eta} \frac{E_U - E^-}{d_U(x)d(x)}.$$

Similarly, we can prove when $x < E^-$. When $E^- \leq x \leq E_U$, we have

$$|\mathcal{X}_E(x) - \mathcal{X}_E * \vartheta_{\tilde{\eta}}(x)| \leq \frac{C\tilde{\eta}}{d_U(x)} + \frac{C\tilde{\eta}}{d(x)} = C\tilde{\eta} \left[\frac{E_U - E^-}{d_U(x)d(x)} + \frac{2\tilde{\eta}}{d_U(x)d(x)} \right],$$

where we use (3.12) of the paper. Therefore, it suffices to show that

$$d_U(x)d(x) \geq \frac{1}{4}\tilde{\eta}(d_U(x) + d(x)) = \frac{1}{4}\tilde{\eta}(E_U - E^- + 2\tilde{\eta}). \tag{S2}$$

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An elementary calculation yields that $d_U(x)d(x) \geq \tilde{\eta}(E_U - E^- + \tilde{\eta})$, which implies (S2). Hence, we conclude our proof. □

For the right-hand side of (S1), when $\min\{d(x), d_U(x)\} \geq t$, it will be bounded by $O(N^{-3\epsilon_0+\epsilon})$; when $\min\{d(x), d_U(x)\} \leq t$, then we must have $\max\{d(x), d_U(x)\} \geq (E_U - E^-)/2$, therefore, it will be bounded by a constant c as $\min\{d(x), d_U(x)\} \geq \tilde{\eta}$. Therefore, by using the above results for the diagonal elements of Q_1, we have

$$
\begin{aligned}
&|\operatorname{Tr} \mathcal{X}_E(Q_1) - \operatorname{Tr} \mathcal{X}_E * \vartheta_{\tilde{\eta}}(Q_1)| \leq C \Big[\operatorname{Tr} f(Q_1) + c\mathcal{N}(E^- - t, E^- + t) + N^{-3\epsilon_0+\epsilon}\mathcal{N}(E^- + t, E_U - t) \\
&\quad + c\mathcal{N}(E_U - t, E_U + t) + N^{-3\epsilon_0+\epsilon}\mathcal{N}(E_U + t, a_{2k-2}) + \sum_{i=1}^{M} \mathcal{X}_E * \vartheta_{\tilde{\eta}}((Q_1)_{ii})\mathbf{1}((Q_1)_{ii} > a_{2k-2})\Big],
\end{aligned} \tag{S3}
$$

where f is defined as

$$
f(x) := \frac{\tilde{\eta}(E_U - E^-)}{d_U(x)d(x)}\mathbf{1}(x \leq E^- - t).
$$

As we assume that $\epsilon < \epsilon_0$, by Assumption 1.2, (2.20) of the paper and the fact $\epsilon_1 < \epsilon$, with $1 - N^{-D_1}$ probability, we have

$$
\mathcal{N}(E_U - t, E_U + t) = 0,\ \mathcal{N}(E_U + t, a_{2k-2}) = 0,\ \mathcal{N}(E^- + t, E_U - t) \leq N^{\epsilon_0}.
$$

On the other hand, when $(Q_1)_{ii} > a_{2k-2}$, by Assumption 1.2 of the paper, we have

$$
\mathcal{X}_E * \vartheta_{\tilde{\eta}}((Q_1)_{ii}) = \tilde{\eta} \int_{(Q_1)_{ii}-E_U}^{(Q_1)_{ii}-E^-} \frac{1}{y^2 + \tilde{\eta}^2} dy \leq \tilde{\eta} \int_{(Q_1)_{ii}-E_U}^{(Q_1)_{ii}-E^-} \frac{1}{y^2} dy \leq \frac{7}{2\tau^2} N^{-4/3+\epsilon-9\epsilon_0},
$$

where τ is defined in Assumption 1.2 of the paper. Hence, we have $\sum_{i=1}^{M} \mathcal{X}_E * \vartheta_{\tilde{\eta}}((Q_1)_{ii})\mathbf{1}((Q_1)_{ii} > a_{2k-2})) \leq CN^{-1/3+\epsilon-9\epsilon_0}$. Therefore, (S3) can be bounded in the following way

$$
|\operatorname{Tr} \mathcal{X}_E(Q_1) - \operatorname{Tr} \mathcal{X}_E * \vartheta_{\tilde{\eta}}(Q_1)| \leq C(\operatorname{Tr} f(Q_1) + \mathcal{N}(E^- - t, E^- + t) + N^{-2\epsilon_0}).
$$

To finish our proof, we need to show that with $1-N^{-D_1}$ probability, $\operatorname{Tr} f(Q_1) \leq N^{-2\epsilon_0}$. By (6.16) of [2], we have

$$
\frac{f(x)}{\tilde{\eta}(E_U - E^-)} \leq C(g * \vartheta_t)(E^- - x),
$$

where $g(y)$ is defined as $g(y) := \frac{1}{y^2+t^2}$. Recall (2.6) and (3.34) of the paper. We have

$$
\frac{1}{N} \operatorname{Tr} \vartheta_t(Q_1 - E^-) = \frac{1}{\pi} \operatorname{Im} m_2(E^- + it).
$$

Hence, we can obtain that

$$
\begin{aligned}
\operatorname{Tr} f(Q_1) &\leq CN\tilde{\eta}(E_U - E^-) \int_{\mathbb{R}} \frac{1}{y^2 + t^2} \operatorname{Im} m_2(E^- - y + it) dy \\
&\leq CN^{1/3+\epsilon}\tilde{\eta} \int_{\mathbb{R}} \frac{1}{y^2 + t^2} [\operatorname{Im} m(E^- - y + it) + \frac{N^{\epsilon_1}}{Nt}] dy,
\end{aligned} \tag{S4}
$$

where we use (2.16) of the paper. It is easy to check that

$$CN^{-1/3+\epsilon+\epsilon_1-9\epsilon_0}\int_{\mathbb{R}}\frac{1}{y^2+t^2}\frac{1}{Nt}dy \leq CN^{-4/3+\epsilon+\epsilon_1-9\epsilon_0}t^{-2}\int_{\mathbb{R}}\frac{t}{t^2+y^2}dy \leq N^{-2\epsilon_0}. \tag{S5}$$

Next, we will use (3.42) of the paper to estimate (S4). When $E^- - y \geq a_{2k-1}$, we have

$$\operatorname{Im} m(E^- - y + it) \leq C\sqrt{t + E^- - y - a_{2k-1}}.$$

Denote $A := \{E^- - y - a_{2k-1} \geq t\}$. Then we have

$$\int_A \frac{\operatorname{Im} m(E^- - y + it)}{y^2+t^2}dy \leq C\int_{\mathbb{R}}\frac{|y|^{1/2}+|E^- - a_{2k-1}|^{1/2}}{y^2+t^2}dy \leq C(\frac{1}{t^{1/2}}+\frac{|E^- - a_{2k-1}|^{1/2}}{y^2+t^2}), \tag{S6}$$

$$\int_{A^c}\frac{\operatorname{Im} m(E^- - y + it)}{y^2+t^2}dy \leq Ct^{-1/2}. \tag{S7}$$

The other case can be treated similarly. Therefore, by (S4), (S5), (S6) and (S7), we have proved $\operatorname{Tr} f(Q_1) \leq N^{-2\epsilon_0}$ holds true with $1 - N^{-D_1}$ probability. Hence, we conclude our proof. □

Next we will follow the approach of [3, Lemma 3.6] to finish the proof of Lemma 3.8. A key observation is that when $s = 0$, we will have a smaller bound but the total number of such terms are $O(N)$ for $x(E)$ and $O(N^2)$ for $y(E)$. And when $s = 1$, we have a larger bound but the number of such terms are $O(1)$. We need to analyze the items with $s = 0, 1$ separately.

Proof of Lemma 3.8. Condition on the variable $s = 0, 1$, we introduce the following decomposition

$$x_s(E) := \frac{N\eta}{\pi}\sum_{k=M+1, \text{ and } \neq \mu,\nu}^{M+N} X_{\mu\nu,k}(E+i\eta)\mathbf{1}(s = \mathbf{1}\left((\{\mu,\nu\}\cap\{\mu_1\}\neq\emptyset)\cup(\{k=\mu_1\})\right)),$$

$$y_s(E) := \frac{\tilde{\eta}}{\pi}\int_{E^-}^{E_U}\sum_k\sum_{\beta\neq k}X_{\beta\beta,k}(E+i\tilde{\eta})dE\mathbf{1}(s=\mathbf{1}((\{\beta=\mu_1\})\cup(\{k=\mu_1\}))).$$

$\Delta x_s, \Delta y_s$ can be defined in the same fashion. Similar to the discussion of (3.64) of the paper, for any E-dependent variable $f \equiv f(E)$ independent of the (i,μ_1)-th entry of X^G, there exist two random variables A_2, A_3, which depend on the randomness only through O, f and the first two moments of $X^G_{i\mu_1}$, for any event Ω, with $1 - N^{-D_1}$ probability, we have

$$\left|\int_I \mathbb{E}_\gamma \Delta x_s(E) f(E) dE - A_2\right|\mathbf{1}(\Omega) \leq ||f\mathbf{1}(\Omega)||_\infty N^{-11/6+C\epsilon_0}N^{-2s/3+t},$$

$$|\mathbb{E}_\gamma \Delta y_s(E) - A_3| \leq N^{-11/6+C\epsilon_0}N^{-2s/3}.$$

In our application, f is usually a function of the entries of R (recall R is independent of V). Next, we use

$$\theta[\int_I x^S q(y^S) dE] = \theta[\int_I (x^R + \Delta x_0 + \Delta x_1) q(y^R + \Delta y_0 + \Delta y_1) dE]. \tag{S8}$$

By (3.60), (3.61) and (3.62) of the paper, it is easy to check that, with $1 - N^{-D_1}$ probability, we have

$$\int_I |\Delta x_s(E)| dE \leq N^{-5/6+C\epsilon_0} N^{-2s/3+t}, \ |\Delta y_s(E)| \leq N^{-5/6+C\epsilon_0} N^{-2s/3}, \tag{S9}$$

$$\int_I |x(E)| dE \leq N^{C\epsilon_0}, \ |y(E)| \leq N^{C\epsilon_0}. \tag{S10}$$

By (S8) and (S9), with $1 - N^{-D_1}$ probability, we have

$$\theta[\int_I x^S q(y^S) dE] = \theta[\int_I x^S (q(y^R) + q'(y^R)(\Delta y_0 + \Delta y_1) + q''(y^R)(\Delta y_0)^2) dE] + o(N^{-2}).$$

Similarly, we have (see (3.44) of [3])

$$\begin{aligned} &\theta[\int_I x^S q(y^S) dE] - \theta[\int_I x^R q(y^R) dE] = \theta'[\int_I x^R q(y^R) dE] \\ &\times [\int_I \left((\Delta x_0 + \Delta x_1) q(y^R) + x^R q'(y^R)(\Delta y_0 + \Delta y_1) + \Delta x_0 q'(y^R) \Delta y_0 + x^R q''(y^R)(\Delta y_0)^2\right) dE] \\ &+ \frac{1}{2}\theta''[\int_I x^R q(y^R) dE][\int_I (\Delta x_0 q(y^R) + x^R q'(y^R) \Delta y_0) dE]^2 + o(N^{-2+t}). \end{aligned} \tag{S11}$$

Now we start dealing with the individual terms on the right-hand side of (S11). Firstly, we consider the terms containing $\Delta x_1, \ \Delta y_1$. Similar to (3.64) of the paper, we can find a random variable A_4, which depends on randomness only through O and the first two moments of $X^G_{i\mu_1}$, such that with $1 - N^{-D_1}$ probability,

$$\left| \mathbb{E}_\gamma \int_I (\Delta x_1 q(y^R) + x^R q'(y^R) \Delta y_1) dE - A_4 \right| = o(N^{-2+t}).$$

Hence, we only need to focus on $\Delta x_0, \ \Delta y_0$. We first observe that

$$\Delta x_0(E) = \mathbf{1}(t=0) \frac{N\eta}{\pi} \sum_{k \neq \mu, \nu, \mu_1} \Delta X_{\mu\nu,k}(z),$$

$$\Delta y_0(E) = \frac{\tilde{\eta}}{\pi} \int_{E^-}^{E_U} \sum_{k \neq \mu_1} \sum_{\beta \neq k, \mu_1} \Delta X_{\beta\beta,k}(E + i\tilde{\eta}) dE.$$

Denote $\Delta x_0^{(k)}(E)$ by the summations of the terms in $\Delta x_0(E)$ containing k items of $X^G_{i\mu_1}$. By (3.46), (3.60) and (3.61) of the paper, it is easy to check that with $1 - N^{-D_1}$ probability,

$$|\Delta x_0^{(3)}| \leq N^{-7/6+C\epsilon_0}, \ |\Delta y_0^{(3)}| \leq N^{-11/6+C\epsilon_0}. \tag{S12}$$

We now decompose $\Delta X_{\mu\nu,k}$ into three parts indexed by the number of $X^G_{i\mu_1}$ they contain. By (3.46), (3.60), (3.61) of the paper and (S12), with $1-N^{-D_1}$ probability, we have

$$\Delta X_{\mu\nu,k} = \Delta X^{(1)}_{\mu\nu,k} + \Delta X^{(2)}_{\mu\nu,k} + \Delta X^{(3)}_{\mu\nu,k} + O(N^{-3+C\epsilon_0}),$$

$$\Delta x_0 = \Delta x_0^{(1)} + \Delta x_0^{(2)} + \Delta x_0^{(3)} + O(N^{-5/3+C\epsilon_0}), \tag{S13}$$

$$\Delta y_0 = \Delta y_0^{(1)} + \Delta y_0^{(2)} + \Delta y_0^{(3)} + O(N^{-7/3+C\epsilon_0}). \tag{S14}$$

Inserting (S13) and (S14) into (S11), similar to the discussion of (3.64) of the paper, we can find a random variable A_5 depending on the randomness only through O and the first two moments of $X^G_{i\mu_1}$, such that with $1-N^{-D_1}$ probability,

$$\begin{aligned}&\mathbb{E}_\gamma\theta[\int_I x^S q(y^S)dE] - \mathbb{E}_\gamma\theta[\int_I x^R q(y^R)dE]\\ &= \mathbb{E}_\gamma\theta'[\int_I x^R q(y^R)dE][\int_I \Delta x_0^{(3)} q(y^R) + x^R q'(y^R)\Delta y_0^{(3)} dE] + A_4 + A_5 + o(N^{-2+t}).\end{aligned}$$

Lemma 3.8 will be proved if we can show

$$\mathbb{E}\theta'[\int_I x^R q(y^R)dE][\int_I \Delta x_0^{(3)} q(y^R) + x^R q'(y^R)\Delta y_0^{(3)} dE] = o(N^{-2}).$$

Due to the similarity, we shall prove

$$\mathbb{E}\theta'[\int_I x^R q(y^R)dE][\int_I \Delta x_0^{(3)} q(y^R)dE] = o(N^{-2}),$$

the other term follows. By (3.3) of the paper and (S10), with $1-N^{-D_1}$ probability, we have $|B^R| := \left|\theta'[\int_I x^R q(y^R)dE]\right| \leq N^{C\epsilon_0}$. Similar to (3.66) of the paper, $\Delta x_0^{(3)}$ is a finite sum of terms of the form

$$\mathbf{1}(t=0)N\eta \sum_{k\neq \mu,\nu,\mu_1} R_{\mu k}(\sigma_i)^{3/2}(X^G_{i\mu_1})^3 \overline{z^{3/2}R_{\nu a_1}R_{b_1 a_2}R_{b_2 a_3}R_{b_3 k}}. \tag{S15}$$

Inserting (S15) into $\int_I \Delta x_0^{(3)} q(y^R)dE$, for some constant $C>0$, we have

$$\left|\mathbb{E}\theta'[\int_I x^R q(y^R)dE][\int_I \Delta x_0^{(3)} q(y^R)dE]\right| \leq N^{-5/6+C\epsilon_0} \max_{k\neq\mu,\nu,\mu_1}\sup_{E\in I}\left|\mathbb{E}B^R R_{\mu k}\overline{R_{\nu\mu_1}R_{ik}}q(y^R)\right| + o(N^{-2}). \tag{S16}$$

Again by (3.60), (3.61) and (3.62) of the paper, it is easy to check that with $1-N^{-D_1}$ probability, for some constant $C>0$, we have

$$|R_{\mu k}\overline{R_{\nu\mu_1}R_{ik}}B^R q(y^R) - S_{\mu k}\overline{S_{\nu\mu_1}S_{ik}}B^S q(y^S)| \leq N^{-4/3+C\epsilon_0}.$$

Therefore, if we can show

$$|\mathbb{E}S_{\mu k}\overline{S_{\nu\mu_1}S_{ik}}B^S q(y^S)| \leq N^{-4/3+C\epsilon_0}, \tag{S17}$$

then by (S16), we finish proving (). The rest leaves to prove (S17). Recall Definition 2.3 and (3.57) of the paper, by [4, Lemma 3.2 and 3.3](or [1, Lemma A.2]), we have the following resolvent identities,

$$S_{\mu\nu}^{(\mu_1)} = S_{\mu\nu} - \frac{S_{\mu\mu_1} S_{\mu_1\nu}}{S_{\mu_1\mu_1}}, \ \mu, \nu \neq \mu_1, \tag{S18}$$

$$S_{\mu\nu} = z S_{\mu\mu} S_{\nu\nu}^{(\mu)} (Y_{\gamma-1}^* S^{(\mu\nu)} Y_{\gamma-1})_{\mu\nu}, \ \mu \neq \nu. \tag{S19}$$

By (3.61), (3.62) of the paper and (S18), it is easy to check that (see (3.72) of [3]),

$$|S_{\mu k} \overline{S_{\nu\mu_1} S_{ik}} B^S q(y^S) - S_{\mu k}^{(\mu_1)} \overline{S_{\nu\mu_1} S_{ik}^{(\mu_1)}} (B^S)^{(\mu_1)} q((y^S)^{(\mu_1)})| \leq N^{-4/3+C\epsilon_0}. \tag{S20}$$

Moreover, by (3.73) of [3], we have

$$S_{\mu k}^{(\mu_1)} \overline{S_{\nu\mu_1} S_{ik}^{(\mu_1)}} (B^S)^{(\mu_1)} q((y^S)^{(\mu_1)}) = (S_{\mu k} \overline{S_{ik}} B^S q(y^S))^{(\mu_1)} \overline{S}_{\nu\mu_1}. \tag{S21}$$

As $t = 0$, by (S19), we have

$$S_{\nu\mu_1} = z m(z) S_{\mu_1\mu_1}^{(\nu)} \sum_{p,q} S_{pq}^{(\nu\mu_1)} (Y_{\gamma-1}^*)_{\nu p} (Y_{\gamma-1})_{q\mu_1} + z(S_{\nu\nu} - m(z)) S_{\mu_1\mu_1}^{(\nu)} \sum_{p,q} S_{pq}^{(\nu\mu_1)} (Y_{\gamma-1}^*)_{\nu p} (Y_{\gamma-1})_{q\mu_1}. \tag{S22}$$

The conditional expectation $\mathbb{E}_\gamma$ applied to the first term of (S22) vanishes; hence its contribution to the expectation of (S21) will vanish. By (2.19) of the paper, with $1 - N^{-D_1}$ probability, we have

$$|S_{\nu\nu} - m(z)| \leq N^{-1/3+C\epsilon_0}. \tag{S23}$$

By the large deviation bound [4, Lemma 3.6], with $1 - N^{-D_1}$ probability, we have

$$\left| \sum_{p,q} S_{pq}^{(\nu\mu_1)} (Y_{\gamma-1}^*)_{\nu p} (Y_{\gamma-1})_{q\mu_1} \right| \leq N^{\epsilon_1} \frac{(\sum_{p,q} |S_{pq}^{(\nu\mu_1)}|^2)^{1/2}}{N}. \tag{S24}$$

By (2.19) of the paper and (S24), with $1 - N^{-D_1}$ probability, we have

$$\left| \sum_{p,q} S_{pq}^{(\nu\mu_1)} (Y_{\gamma-1}^*)_{\nu p} (Y_{\gamma-1})_{q\mu_1} \right| \leq N^{-1/3+C\epsilon_0}. \tag{S25}$$

Therefore, inserting (S23) and (S25) into (S21), by (2.19) of the paper, we have

$$|\mathbb{E} S_{\mu k}^{(\mu_1)} \overline{S_{\nu\mu_1} S_{ik}^{(\mu_1)}} (B^S)^{(\mu_1)} q((y^S)^{(\mu_1)})| \leq N^{-4/3+C\epsilon_0}.$$

Combine with (S20), we conclude our proof. □

Proof of Lemma 4.4. It is easy to check that with $1-N^{-D_1}$ probability, (3.39) of the paper still holds true. Therefore, it remains to prove the following result

$$\mathbb{E}^V\theta[\frac{N}{\pi}\int_I \tilde{G}_{\mu\nu}(E+i\eta)q(\operatorname{Tr}\mathcal{X}_E(Q_1))]-\mathbb{E}^V\theta[\frac{N}{\pi}\int_I \tilde{G}_{\mu\nu}(E+i\eta)q(\operatorname{Tr}f_E(Q_1))dE]=o(1). \tag{S26}$$

We first observe that for any $x\in\mathbb{R}$, we have

$$|\mathcal{X}_E(x)-f_E(x)|=\begin{cases}0, & x\in[E^-,E_U]\cup(-\infty,E^--\eta_d)\cup(E_U+\eta_d,+\infty);\\ |f_E(x)|, & x\in[E^--\eta_d,E^-)\cup(E_U,E_U+\eta_d].\end{cases}$$

Therefore, we have

$$|\operatorname{Tr}\mathcal{X}_E(Q_1)-\operatorname{Tr}f_E(Q_1)|\leq\max_x|f_E(x)|\left(\mathcal{N}(E^--\eta_d,E^-)+\mathcal{N}(E_U,E_U+\eta_d)\right).$$

By Lemma 2.3 of the paper, the definition of η_d and a similar argument to (3.44) of the paper, we can finish the proof of (S26). □

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